

ERRATUM TO: LATTICE POINT METHODS FOR COMBINATORIAL GAMES

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This note corrects Definition 6.3, Proposition 6.7, and Section 7, as specified below.

Proposition 6.7 is false for the notion of squarefree game in Definition 6.3, i.e., the equivalent conditions in Proposition 6.2. Henceforth those equivalent conditions define *weakly squarefree games*. For example, the game on \mathbb{N}^3 with rule set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0)\}$ is weakly squarefree but its P-positions are easily shown not to satisfy Proposition 6.7.

The intended notion of *squarefree game* is any game satisfying the conditions in the following, whose parts mirror Proposition 6.2 as closely as possible.

Proposition 1. *For a rule set Γ , the following are equivalent.*

1. *For each $\gamma \in \Gamma$, and $p, q \in \mathbb{N}^d$, if $p + q - \gamma \in \mathbb{N}^d$ then $p - \gamma \in \mathbb{N}^d$ or $q - \gamma \in \mathbb{N}^d$.*
2. *(There is no appropriate analogue of condition 2 in Proposition 6.2.)*
3. *Each $\gamma \in \Gamma$ has at most one positive entry, and that entry is at most 1.*
4. *The positive part γ_+ is a 0-1 vector with at most one 1, for all $\gamma \in \Gamma$.*
5. *Each move decreases the number of heaps of exactly one size, and the amount of the decrease is 1.*

Proof. $1 \Rightarrow 3$: Fix $\gamma = (\gamma_1, \dots, \gamma_d) \in \Gamma$. First we show that the maximum entry of γ is at most 1. Let $M = \max\{\gamma_1, \dots, \gamma_d\}$, and let $p = \lceil \frac{M}{2} \rceil \mathbb{1}$ where $\mathbb{1}$ is the vector with all entries equal to 1. Then the minimum of the entries of $2p - \gamma$ is 1 for odd M and 0 for even M , and hence $2p - \gamma \in \mathbb{N}^d$. However, the minimum of the entries of $p - \gamma$ is $\lceil \frac{M}{2} \rceil - M$ which is negative if $M > 1$. But $p - \gamma \in \mathbb{N}^d$ by hypothesis, so $M \leq 1$. Next we show that at most one entry equals 1. Suppose that more than one entry is 1, say $\gamma_i = \gamma_j = 1$ where $i \neq j$. For each $k = 1, \dots, d$, let e_k be the k -th basis vector. Let $p = \gamma \vee \mathbf{0} - e_j$ and set $q = e_j$. Then $p + q = \gamma \vee \mathbf{0}$ so $p + q - \gamma \in \mathbb{N}^d$, but $(p - \gamma)_j = -1$ and $(q - \gamma)_i = -1$.

$3 \Rightarrow 1$: Fix $\gamma \in \Gamma$. Then γ has at least one entry $\gamma_i = 1$, and therefore exactly one, because the real cone $\mathbb{R}_+ \Gamma$ contains \mathbb{R}_+^d and is pointed. If $p, q \in \mathbb{N}^d$ with $p + q - \gamma \in \mathbb{N}^d$, then $p_i + q_i \geq 1$, whence at least one of p_i and q_i is ≥ 1 . Say $p_i \geq 1$. Then $p - \gamma \in \mathbb{N}^d$ because $\gamma_j \leq 0$ for all $j \neq i$.

$3 \Leftrightarrow 4$: This is straightforward.

$4 \Leftrightarrow 5$: In the notation of Examples 2.1 and 2.5, condition 5 is the translation of condition 4 into the language of heaps. \square

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Next we verify that Proposition 6.7 does indeed hold for the corrected notion of squarefree game.

Proposition 2. *If $p \in \mathcal{B}$, then $p \in \mathcal{P} \Leftrightarrow p \cong \mathbf{0}$.*

Proof. Observe that $\mathbf{0} \in \mathcal{P}$; consequently, $p \cong \mathbf{0} \Rightarrow p \in \mathcal{P}$. Therefore it remains only to prove the other direction, namely $p \in \mathcal{P} \Rightarrow p \cong \mathbf{0}$. This has two parts: $p + q \in \mathcal{P}$ whenever $p, q \in \mathcal{P}$, and $p + q \in \mathcal{N}$ whenever $q \in \mathcal{N}$. However, the second follows from the first, because $q \in \mathcal{N} \Rightarrow q - \gamma \in \mathcal{P}$ for some $\gamma \in \Gamma$, so adding the two P-positions p and $q - \gamma$ always yields another P-position $p + q - \gamma$, whence $p + q \in \mathcal{N}$.

To finish the proof, we show that $p, q \in \mathcal{P} \Rightarrow p + q \in \mathcal{P}$ by induction. Clearly $p + q \in \mathcal{P}$ if $p = q = \mathbf{0}$. Therefore assume $p, q \in \mathcal{P}$ with $p \succ \mathbf{0}$ or $q \succ \mathbf{0}$, and inductively assume that $\hat{p} \in \mathcal{P} \Rightarrow \hat{p} + q \in \mathcal{P}$ for all $\hat{p} \prec p$ and $p + \hat{q} \in \mathcal{P} \Rightarrow p + \hat{q} \in \mathcal{P}$ for all $\hat{q} \prec q$. Fix $\gamma \in \Gamma$ such that $p + q - \gamma \in \mathbb{N}^d$. (Such a γ exists by Proposition 1, because the tangent cone axiom implies the existence of a move whose only negative coordinate is -1 and occurs where $p + q$ is positive; but even if no such γ existed we would already be done anyway, because then $p + q \in \mathcal{P}$ by definition.) By Proposition 1, either $p - \gamma \in \mathbb{N}^d$ or $q - \gamma \in \mathbb{N}^d$. Suppose $p - \gamma \in \mathbb{N}^d$. Then $p - \gamma \in \mathcal{N}$, so $p - \gamma - \gamma' \in \mathcal{P}$ for some $\gamma' \in \Gamma$. By our induction hypothesis, $(p - \gamma - \gamma') + q = (p + q - \gamma) - \gamma' \in \mathcal{P}$, so $p + q - \gamma \in \mathcal{N}$. If $q - \gamma \in \mathbb{N}^d$, then a similar argument still yields $p + q - \gamma \in \mathcal{N}$. Since γ was arbitrary, $p + q \in \mathcal{P}$. \square

The proof of the algorithm in Section 7 is incorrect, although the existence of the algorithm is still true. We offer a simpler proof which obviates Section 7, and we also provide analysis of the complexity of the algorithm.

Theorem 3. *There is an algorithm for computing \mathcal{P} for a squarefree game in normal play that runs in $O(2^d|\Gamma|)$ time and requires $O(2^d)$ space.*

Proof. By Theorem 6.11 it suffices to compute $\mathcal{P}_0 = \mathcal{P} \cap \{0, 1\}^d$. Let the *outcome* of a position $\mathbf{p} \in \mathcal{B}$ be P if \mathbf{p} is a P-position, and N otherwise. If we use TRUE to encode P-positions and FALSE to encode N-positions, then the outcome of a position $\mathbf{p} \in \mathcal{B}$ is the logical NOR of its legal options. Therefore, we can use a dynamic programming approach by recursively computing the outcomes of all the positions in $\{0, 1\}^d$ while storing the results in memory so that the outcome of each position need only be computed once. Furthermore, if a legal option \mathbf{p}' of \mathbf{p} lies outside $\{0, 1\}^d$, then by Theorem 6.11 the outcome of \mathbf{p}' is the same if we take its coordinates modulo 2. Since there are 2^d positions to compute and each position is computed exactly once, which requires looking at the NOR of at most $|\Gamma|$ outcomes, the algorithm runs in $O(2^d|\Gamma|)$ time and requires $O(2^d)$ space. \square

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